# Approximate Graph Propagation


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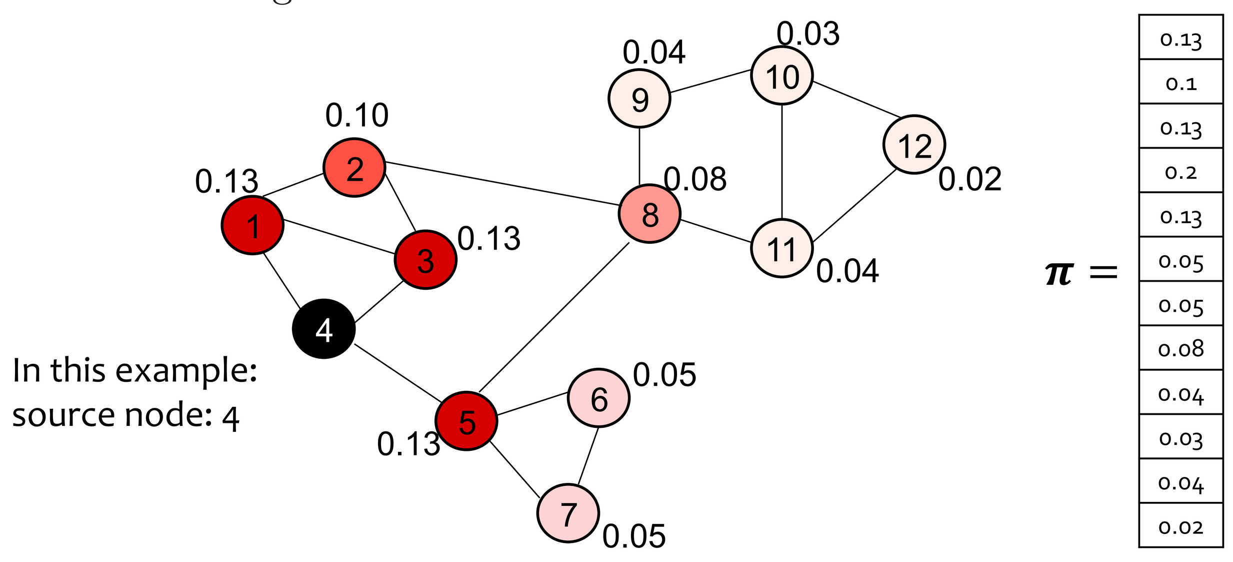
## 1. Motivations

- **Node proximity:**
  - ◇ Node proximity measures the relative importance of nodes on the graph with respect to the given source node.
  - ◇ **Problem definition:** The node proximity query computes the node proximity vector $\boldsymbol{\pi}$, where $\boldsymbol{\pi}(v)$ denotes node $v$'s proximity w.r.t the given source node.




- **Popular node proximities**:

| Node proximity | Propagation equation |
|---|---|
| $L$-hop transition probability | $\boldsymbol{\pi} = \left(\mathbf{AD}^{-1}\right)^{L} \cdot \boldsymbol{e}_s$ |
| PageRank [1] | $\boldsymbol{\pi} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i \cdot \left(\mathbf{AD}^{-1}\right)^i \cdot \frac{\mathbf{1}}{n}$ |
| Personalized PageRank [1] | $\boldsymbol{\pi} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i \cdot \left(\mathbf{AD}^{-1}\right)^i \cdot \boldsymbol{e}_s$ |
| Single-target PPR [2] | $\boldsymbol{\pi} = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i \cdot \left(\mathbf{D}^{-1}\mathbf{A}\right)^i \cdot \boldsymbol{e}_t$ |
| Heat Kernel PageRank [3] | $\boldsymbol{\pi} = \sum_{i=0}^{\infty} e^{-t} \cdot \frac{t^i}{i!} \cdot \left(\mathbf{AD}^{-1}\right)^i \cdot \boldsymbol{e}_s$ |
| Katz [4] | $\boldsymbol{\pi} = \sum_{i=0}^{\infty} \beta^i \cdot \mathbf{A}^i \cdot \boldsymbol{e}_s$ |

- **Feature propagation in GNN**:

| GNN models | Propagation equation |
|---|---|
| SGC [5] | $\boldsymbol{\pi} = \left(\mathbf{D}^{-1/2}\mathbf{AD}^{-1/2}\right)^{L} \cdot \boldsymbol{x}$ |
| APPNP [6] | $\boldsymbol{\pi} = \sum_{i=0}^{L} \alpha(1-\alpha)^i \cdot \left(\mathbf{D}^{-1/2}\mathbf{AD}^{-1/2}\right)^i \cdot \boldsymbol{x}$ |
| GDC [7] | $\boldsymbol{\pi} = \sum_{i=0}^{L} e^{-t} \cdot \frac{t^i}{i!} \cdot \left(\mathbf{D}^{-1/2}\mathbf{AD}^{-1/2}\right)^i \cdot \boldsymbol{x}$ |

- **Two major questions:**
  - ◇ Is there a ***generalized graph propagation equation***?
  - ◇ Is there a ***universal algorithm*** that computes the approximate graph propagation with **near optimal cost**?

## 2. Contributions

- **We propose a Generalized Graph Propagation Equation:**

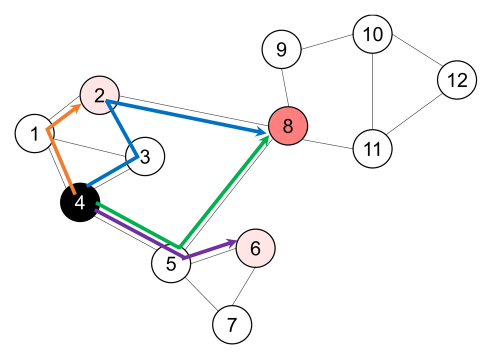
$$\boldsymbol{\pi} = \sum_{i=0}^{\infty} \boldsymbol{w_i} \cdot \left(\mathbf{D}^{-a}\boldsymbol{A}\mathbf{D}^{-b}\right)^{i} \cdot \boldsymbol{x}$$

| Node proximities / GNN models | weighted sequence $w_i$ | propagation matrix | graph signal $\boldsymbol{x}$ |
|---|---|---|---|
| transition probability/ SGC | $w_L = 1$, $w_i = 0\ (i \neq L)$ | $\mathrm{AD}^{-1}$/ $\mathrm{D}^{-1/2}\mathrm{AD}^{-1/2}$ | $\boldsymbol{e_s}$ / the feature vector $\boldsymbol{x}$ |
| PPR / APPNP | $\alpha(1-\alpha)^i$ | $\mathrm{AD}^{-1}$/ $\mathrm{D}^{-1/2}\mathrm{AD}^{-1/2}$ | $\boldsymbol{e_s}$ / the feature vector $\boldsymbol{x}$ |
| HKPR / GDC | $e^{-t} \cdot \frac{t^i}{i!}$ | $\mathrm{AD}^{-1}$/ $\mathrm{D}^{-1/2}\mathrm{AD}^{-1/2}$ | $\boldsymbol{e_s}$ / the feature vector $\boldsymbol{x}$ |
| Katz | $\beta^i$ | A | $\boldsymbol{e_s}$ |

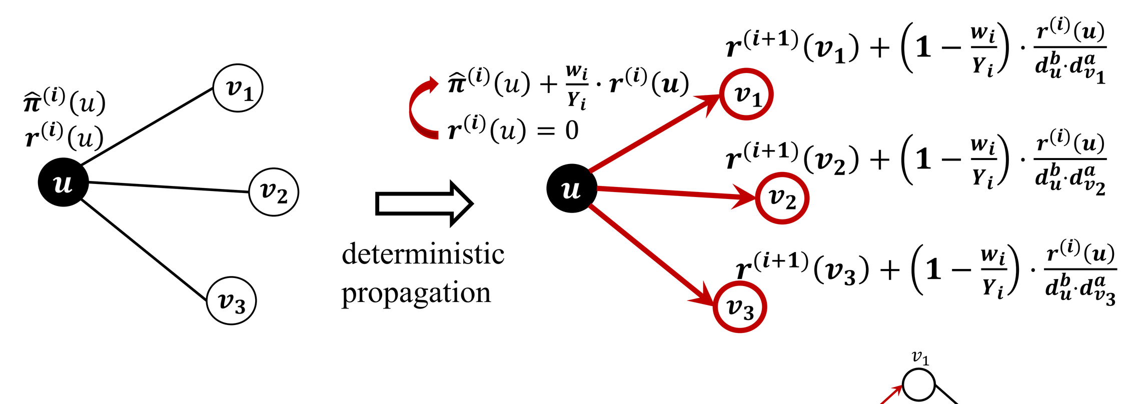
- We propose **Approximate Graph Propagation (AGP)**, **a unified randomized algorithm** that computes various node proximities and GNN models in **near optimal time**.

## 3. Previous Work

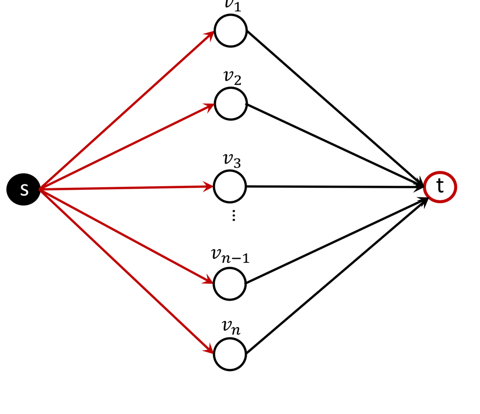
- **Monte-Carlo method:**
  - ◇ Generate multiple random walks from the source node
  - ◇ $\hat{\boldsymbol{\pi}}(v) = \frac{\#\text{ of walks terminate at } v}{\text{total }\#\text{ of walks}}$.
  - ◇ **Drawbacks:**
    - Multiple walks needed.
    - Don't support Katz ($a + b \neq 1$).



- **Deterministic propagation [8, 9]:**
  - ◇ residue $\boldsymbol{r}^{(i)}(u)$: the probability mass to be propagated to node $v$ in level $i$.
  - ◇ reserve $\hat{\boldsymbol{\pi}}^{(i)}(u)$: the probability mass stays at node $v$ in level $i$.




  - ◇ **Drawbacks:**
    In some bad cases, to derive $\hat{\boldsymbol{\pi}}(t)$:
    - Deterministic propagation: $O(n)$
    - Monte Carlo method: $O(1)$.

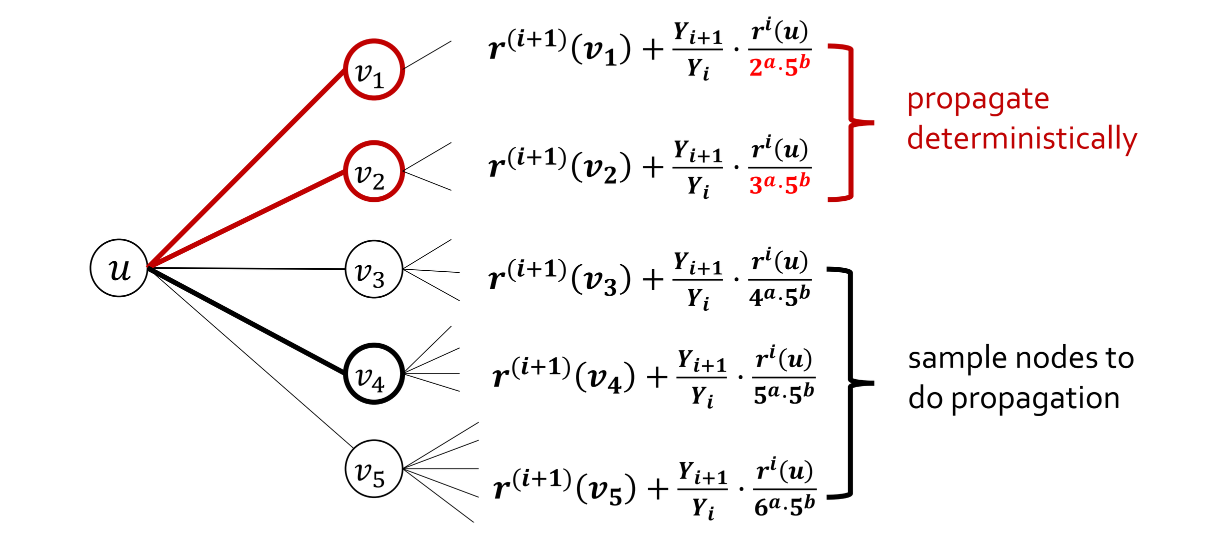


## 4. Algorithms

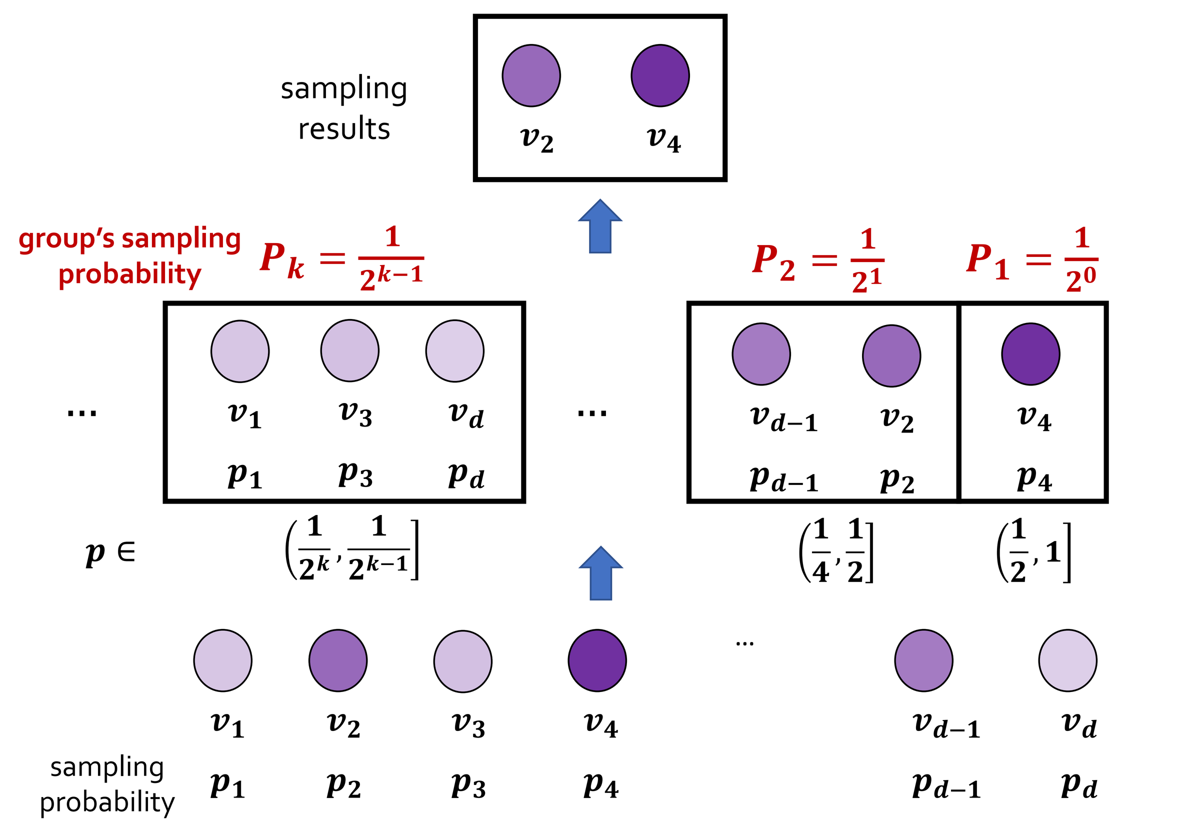
- **AGP:** combine the strengths of Monte-Carlo method and the deterministic propagation
  - ◇ In the propagation from node $u$ at level $i$ to $v$ at level $i+1$, the increment of $v$'s residue is:

$$\boldsymbol{r}^{i+1}(v) \leftarrow \boldsymbol{r}^{i+1}(v) + \frac{Y_{i+1}}{Y_i} \cdot \frac{\boldsymbol{r}^i(u)}{d_u^b \cdot d_v^a}.$$

  - ◇ **Pre-sorting adjacency list by degrees:**




  - ◇ **Randomized propagation with subset sampling:**
    - In each group, sample according to the binary distribution with the groups' sampling probability.
    - Reject the selected nodes w.p. $\frac{p_{individual}}{p_{group}}$ for unbiasedness.

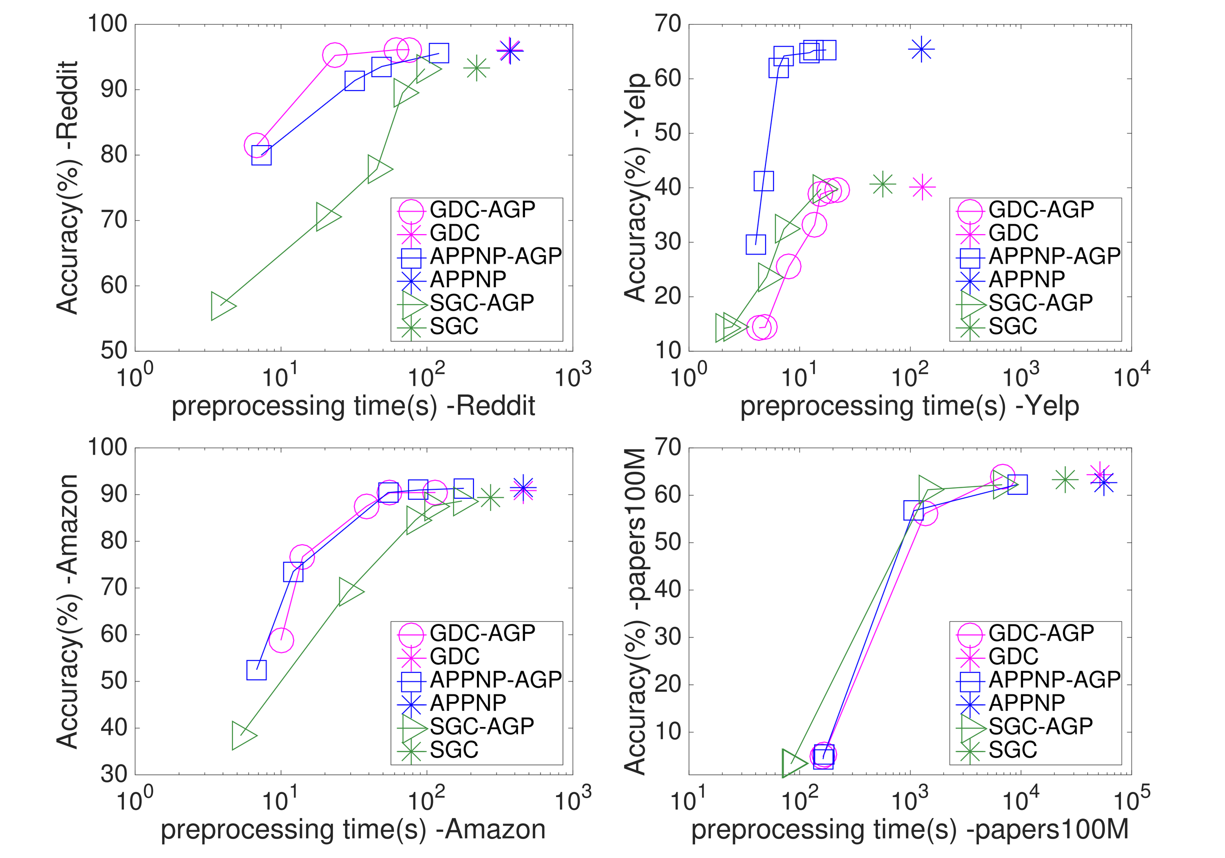



## 5. Experiments

- **Node Classification with GNN**

| Data Set | $n$ | $m$ | $d$ | Classes | Label % |
|---|---|---|---|---|---|
| Reddit | 232,965 | 114,615,892 | 602 | 41 | 0.0035 |
| Yelp | 716,847 | 6,977,410 | 300 | 100 | 0.7500 |
| Amazon | 2,449,029 | 61,859,140 | 100 | 47 | 0.7000 |
| Papers100M | 111,059,956 | 1,615,685,872 | 128 | 172 | 0.0109 |

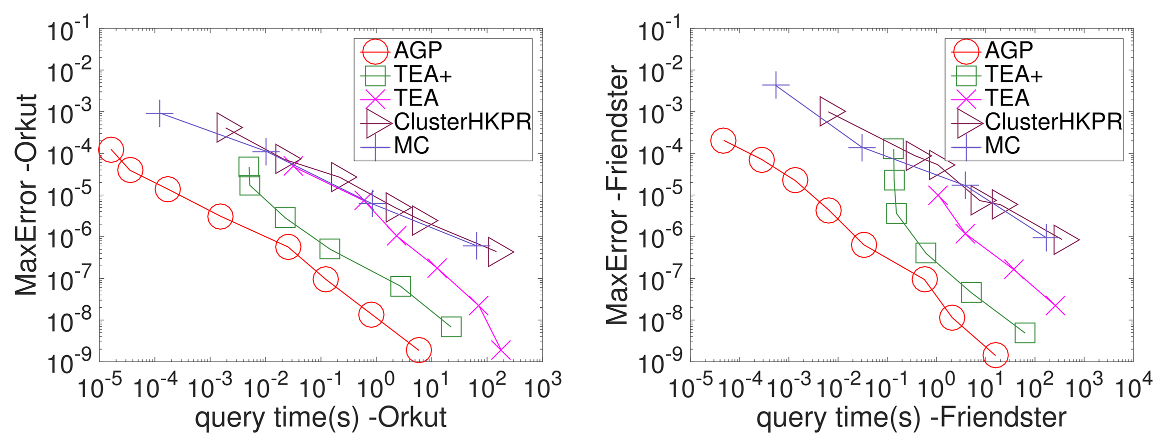
  - ◇ **Tradeoffs between** ***Accuracy*** **(%) and propagation time:**



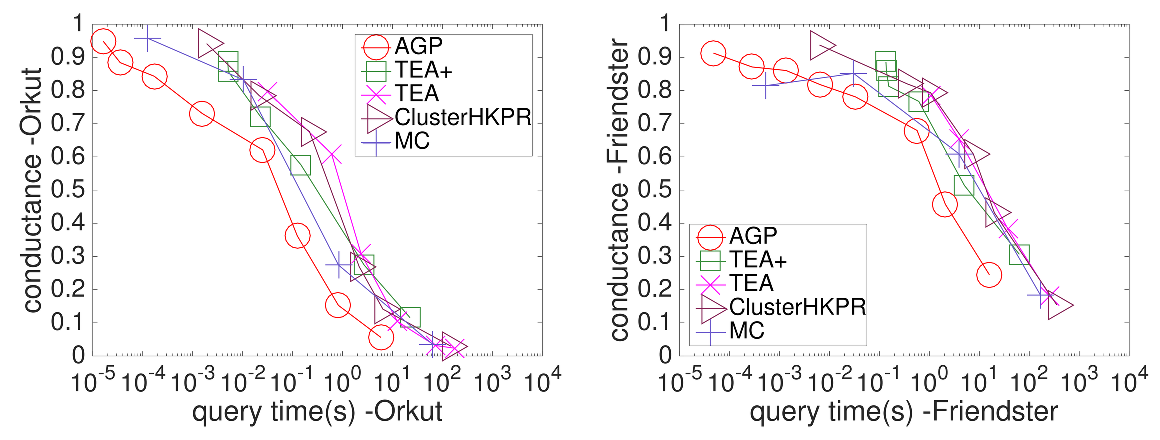

- **Local Clustering with HKPR**

| Dataset | Type | $n$ | $m$ |
|---|---|---|---|
| Orkut-Links | undirected | 3,072,441 | 234,369,798 |
| Friendster | undirected | 68,349,466 | 3,623,698,684 |

  - ◇ **Tradeoffs between** ***MaxError*** **and query time:**




  - ◇ **Tradeoffs between** ***conductance*** **and query time:**

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